

New AdS solitons and brane worlds with compact extra-dimensions

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ABSTRACT: We construct new static, asymptotically AdS solutions where the conformal infinity is the product of Minkowski spacetime M_n and a sphere S^m . Both globally regular, soliton-type solutions and black hole solutions are considered. The black holes can be viewed as natural AdS generalizations of the Schwarzschild black branes in Kaluza-Klein theory. The solitons provide new brane-world models with compact extra-dimensions. Different from the Randall-Sundrum single-brane scenario, a Schwarzschild black hole on the Ricci flat part of these branes does not lead to a naked singularity in the bulk.

KEYWORDS: AdS solitons, black branes, numerical solutions.

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1. Introduction

In the last decade a tremendous amount of interest has been focused on asymptotically Anti-de Sitter (AdS) spacetimes. This interest is mainly motivated by the proposed correspondence between physical effects associated with gravitating fields propagating in AdS spacetime and those of a conformal field theory (CFT) on the boundary of AdS [1, 2].

However, this correspondence does not impose any constraint on the way of approaching the boundary of AdS spacetime. Depending on the choice of 'radial' coordinate defining the family of surfaces which approach the boundary, the slices of constant radius can have different geometries or even different topologies. For example, as discussed in [3], the maximally symmetric (euclideanized) AdS_d has a wide variety of possible boundary geometries, *e.g.* $S^1 \times S^{d-2}$, $S^1 \times R^{d-2}$, $S^1 \times H^{d-2}$, S^{d-1} , R^{d-1} , H^{d-1} or even $S^m \times H^{d-m-1}$ (with S^m , R^m and H^m the m -dimensional sphere, Euclidean plane and hyperbolic space, respectively).

Of course, the allowed possibilities increase if instead of maximally symmetric AdS one takes asymptotically (locally) AdS solutions of the Einstein equations with negative cosmological constant Λ . Although a variety of solutions with nontrivial boundary structure were investigated¹, the issue of constructing bulk solutions compatible with a given boundary metric is not yet fully explored. The main obstacle here seems to be the extreme difficulty to solve the field equations (and thus the generic absence of exact solutions) for more complicated boundary metrics.

Moreover, as expected, there is no unique way to construct a bulk metric for a given boundary structure. For example, a black hole has the same leading order expansion as the AdS background, while bubble solutions may also be relevant. These aspects are nicely illustrated by the following example. Given the boundary geometry $R_t \times S^1 \times R^{d-3}$, there are at least three solutions of the Einstein equations. First, there is the AdS_d spacetime written in Poincaré coordinates (with suitable identifications), then a topological black hole spacetime with a Ricci flat horizon and finally the AdS soliton. The latter is a globally regular solution of the Einstein equations with $\Lambda < 0$ which was found by Horowitz and Myers in [6] and has played an important role in conceptual developments in general relativity and in AdS/CFT. It has been conjectured by Horowitz and Myers that for a $R_t \times S^1 \times R^{d-3}$ boundary topology, the AdS soliton is the minimum energy (perturbatively stable) solution of the Einstein equations (its energy is lower than that of the AdS spacetime itself). Moreover, regarding the AdS soliton as a reference background, Surya, Schleich and Witt found that a phase transition occurs between the Ricci flat AdS black hole (where at least one of the horizon coordinates is taken to be compact) and the thermal AdS soliton [7]. However, there is no phase transition for AdS black holes with Ricci flat horizons when the zero mass black hole is taken as the thermal background.

The main purpose of this work is to present evidence for the existence of a class of generalizations of the AdS soliton, with the S^1 circle there replaced by a sphere S^m . As in the $m = 1$ case, the boundary metric has also a Ricci flat part, which can be taken to be Minkowski spacetime in n -dimensions, M_n . However, for $m > 1$, there is no foliation of AdS spacetime leading to a boundary metric $M_n \times S^m$, since this configuration is not maximally symmetric. This leads to a rather different set of features of the $m > 1$ configurations as compared to the Horowitz-Myers soliton, in particular the absence of free parameters in the solution². Moreover, we argue that the $m > 1$ solitons emerge as zero event horizon radius limit of some black hole solutions with the same conformal boundary at infinity. These black holes have a horizon topology $R^{n-1} \times S^m$.

In this work we examine the general properties of both solitons and black holes with a boundary metric $M_n \times S^m$ and compute their global charges by using a counterterm prescription. For our ansatz, the Einstein equations reduce to ordinary differential equations and, although we could not find an exact solution, it is straightforward to integrate them numerically by matching the near origin/horizon expansion to their Fefferman-Graham expansion

¹This includes even boundary metrics which are not globally hyperbolic, see *e.g.* [4], [5].

²This resembles the case of some field theory solitons in a flat spacetime background, see *e.g.* [8], [9], [10].

[11]. Given the presence of a compact sphere S^m in the metric, the black holes share many properties of the well-known Schwarzschild-AdS black holes with spherical topology of the horizon, in particular the existence of two branches of solutions with different thermodynamical properties.

These solutions would provide the gravity dual for conformal gauge theories defined on a fixed $M_n \times S^m$ background, the black holes accounting for finite temperature effects. Apart from that, the existence of a flat sector of the solitons' metric suggests a possible role of these solutions in a brane world context. In this work we demonstrate that, by applying a cutting and pasting procedure analogous to that used in the Randall-Sundrum (RS) scenario [12], the solitons yield new brane world models. Apart from a Ricci flat part, which in the simplest case has a Poincaré symmetry, the brane metric has an m -dimensional spherical internal space of constant, fixed radius. An interesting feature here is that, different from the original RS case, a Schwarzschild black hole on the Ricci flat part of the brane does not lead to any pathology in the bulk. Of course, there is price to pay for that. The presence of two different sectors of the brane world metric, with different topologies, leads to an anisotropic stress tensor on the brane. Thus, apart from a brane tension (which is present also in the RS model) one has to assume the existence of some extra matter fields, which we take to be a topological soliton confined on the m -dimensional sphere.

Our paper is structured as follows: the AdS solitons are discussed in the next section, where we present both analytical and numerical arguments for their existence. The basic properties of a brane world model based on these solutions are discussed in Section 3. In Section 4 we present the results obtained by numerical calculations in the case of the black hole solutions. We give our conclusions and remarks in the final Section.

2. New AdS solitons

2.1 The action and metric ansatz

We start with the following action in d -spacetime dimensions

$$I_0 = \frac{1}{16\pi G} \int_{\mathcal{M}} d^d x \sqrt{-g} (R - 2\Lambda) - \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^{d-1} x \sqrt{-\gamma} K, \quad (2.1)$$

where G is the gravitational constant in d dimensions and $\Lambda = -(d-1)(d-2)/(2\ell^2)$ is the cosmological constant. Here \mathcal{M} is a d -dimensional manifold with metric $g_{\mu\nu}$, K is the trace of the extrinsic curvature $K_{ab} = -\gamma_a^c \nabla_c n_b$ of the boundary $\partial\mathcal{M}$ with unit normal n^a and induced metric γ_{ab} .

The classical equations of motion are derived by setting the variations of the action (2.1) to zero,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \frac{(d-1)(d-2)}{2\ell^2} g_{\mu\nu} = 0. \quad (2.2)$$

We are interested in solutions of the Einstein equations whose spacelike infinity is the product of an n -dimensional Ricci flat space and an m -dimensional sphere. In the simplest

case, such solutions can be described within the following metric ansatz:

$$ds^2 = \frac{dr^2}{f(r)} + a(r)d\Sigma_n^2 + P^2(r)d\omega_m^2, \quad (2.3)$$

where $n+m+1 = d$. In (2.3), $d\Sigma_n^2$ denotes an arbitrary Ricci flat metric; for most of this work we shall restrict ourselves to the simplest case of a Minkowski metric $d\Sigma_n^2 = \sum_{i,j=1}^{n-1} \delta_{i,j} dx^i dx^j - dt^2$. Also, $d\omega_m^2$ is the unit metric on S^m and r is a radial coordinate, with $r_i \leq r < \infty$. We shall suppose that, as $r \rightarrow r_i$, the proper area of the S^m -sphere goes to zero, while that of the flat subspace remains nonzero, $a(r_i) > 0$. In the asymptotic region $r \rightarrow \infty$, the solutions are supposed to be locally asymptotically AdS.

The functions P, a and f are solutions of the differential equations

$$\begin{aligned} P'' + \frac{P'f'}{2f} + \frac{1}{2(d-2)} \left((d-m-2)(d-m-1) \frac{a'P'}{a} + (2d-m-4)(m-1) \frac{P'^2}{P} \right. \\ \left. - (d-m-2)(d-m-1) \frac{Pa'^2}{4a^2} - \frac{(2d-m-4)(m-1)}{fP} + \frac{2\Lambda P}{f} \right) = 0, \\ a'' + \frac{a'f'}{2f} + \frac{1}{(d-2)} \left(-\frac{(m-1)maP'^2}{P^2} + \frac{((d-7)d-m^2+m+10)a'^2}{4a} \right. \\ \left. + \frac{(m-1)ma'P'}{P} + \frac{(m-1)ma}{fP^2} + \frac{2\Lambda a}{f} \right) = 0, \\ \frac{(d-m-2)(d-m-1)P^2a'^2}{4ma^2} + \frac{(d-m-1)Pa'P'}{a} + (m-1)P'^2 - \frac{(m-1)}{f} + \frac{2\Lambda P^2}{mf} = 0, \end{aligned} \quad (2.4)$$

where a prime denotes a derivative with respect to r . Note also that the ansatz (2.3) still has some gauge freedom which can be used to fix one of the functions a, f or P , the corresponding equation becoming a constraint.

2.2 Known solutions

The cases $m = 0$ and $n = 0$ are rather special, as well as $n = 1$. For $m = 0$ (*i.e.* no compact directions), the solution of (2.2) has $a(r) = f(r) = r^2/\ell^2$, and corresponds to a maximally symmetric AdS_d spacetime [3]. By defining $z = -\ell \log r/\ell$, the solution becomes AdS in horospherical coordinates,

$$ds^2 = dz^2 + e^{-2z/\ell} d\Sigma_n^2, \quad (2.5)$$

with a R^{d-1} boundary at infinity³. The solution for the case $n = 0$ (*i.e.* no flat directions) has $f(r) = r^2/\ell^2 + 1$, $P(r) = r$, which is a parametrization of the Euclidean AdS_d with a S^{d-1} boundary [3]. The global AdS spacetime can be written also within the ansatz (2.3) for $n = 1$ and an arbitrary $m \geq 1$, and has $a(r) = f(r) = r^2/\ell^2 + 1$, $P(r) = r$, the boundary metric in this case being $R \times S^{d-2}$.

³Note that (2.5) is basically the main form used in RS brane world models.

However, the main known solution of interest in the context of our work has $m = 1$, $n \geq 3$ and corresponds to the Horowitz and Myers AdS soliton [6]. The most convenient choice for the metric gauge here is $a(r) = r^2$, which results in the following solution of the equations (2.4):

$$f(r) = P^2(r) = \frac{r^2}{\ell^2} - M_0 \left(\frac{\ell}{r}\right)^{d-3}, \quad (2.6)$$

$M_0 > 0$ being an arbitrary parameter. The range of the radial coordinate is restricted in this case to $r \geq r_i = \ell M_0^{1/(d-1)}$, where $P(r_i) = 0$. In fact, $r = r_i$ represents the origin of the coordinate system and is a 'bolt'. The solution is free of conical singularities for a periodicity

$$\beta = \frac{4\pi\ell}{(d-1)M_0^{1/(d-1)}} \quad (2.7)$$

of the compact S^1 -coordinate. The mass of the AdS soliton is $\mathcal{M} = -M_0\beta\ell^{d-3}V_x/(16\pi G)$, which is lower than AdS itself [6] (V_x is the coordinate volume of the surface parametrized by x^i (with $i = 1, \dots, n-1$)). This result has found close agreement with the negative Casimir energy of non-supersymmetric field theory on $S^1 \times R^n$ [6].

Another particular case which has been already studied in the literature is $n = 2$ (*i.e.* $d\Sigma_2^2 = dx^2 - dt^2$) and an arbitrary dimension of the sphere, $m > 1$. Different from the cases above, no analytic solution is available here and one has to integrate the equations (2.4) numerically. However, the main emphasis in the literature was on the black hole generalizations of these solitons, which were interpreted in [13], [14] as describing the AdS counterparts of the black strings in the $\Lambda = 0$ Kaluza-Klein theory. They present a number of interesting features and various extensions were studied subsequently in [15]-[19].

2.3 New $m > 1$ solutions: the asymptotic expansion

The $m > 1$ solutions can be thought of as higher dimensional generalizations of the $m = 1$ AdS soliton, with the S^1 direction replaced by a sphere. Unfortunately, we could not find an exact solution in this case. However, one can analyse the properties of these configurations by using a combination of analytical and numerical methods, which is enough for most purposes.

For $m > 1$, we have found it convenient to fix the metric gauge⁴ by taking $P(r) = r$, such that $r_i = 0$ and thus the range⁵ of the radial coordinate is $0 \leq r < \infty$. It may be interesting to note that by taking $a = e^{2q}$, the system (2.4) reduces in this case to a single first order nonlinear differential equation for $A = q'$ (the function f can be expressed in terms of a and a')

$$A' + \psi_3 A^3 + \psi_2 A^2 + \psi_1 A + \psi_0 = 0, \quad (2.8)$$

⁴This choice has also been used in the study [14] of the $n = 2$ solitons and black string solutions.

⁵In Section 3 we will cut off the radial extent by inserting a brane at some finite r_0 .

where

$$\begin{aligned}\psi_0 &= \frac{2\Lambda m(m-1)}{(d-2)(m(m-1)-2\Lambda r^2)}, \quad \psi_1 = \frac{m}{r} + \frac{2\Lambda r(2-3m(m+1)+d(3m-1))}{(d-2)(m(m-1)-2\Lambda r^2)}, \\ \psi_2 &= \frac{2(d-m-1)((d-2)(m-1)m+\Lambda r^2(d-3m-2))}{(d-2)(m(m-1)-2\Lambda r^2)}, \\ \psi_3 &= \frac{r(d-m-2)(d-m-1)((d-2)(m-1)-2\Lambda r^2)}{(d-2)(m(m-1)-2\Lambda r^2)},\end{aligned}\tag{2.9}$$

whose solutions we could not find in closed form, however.

Smoothness at $r = 0$ requires that $a(r), f(r)$ have there a Taylor series consisting of even powers of r only, with $a(0) > 0$ and $f(0) = 1$. To order r^4 , the small r solution reads

$$\begin{aligned}a(r) &= a(0) \left(1 - \frac{2\Lambda}{(m+1)(n+n-1)} r^2 + \frac{4\Lambda^2(n-1)}{m(m+1)^2(m+3)(m+n-1)} r^4 \right) + O(r)^6, \\ f(r) &= 1 - \frac{2\Lambda}{m(m+1)(m+n-1)} r^2 + \frac{4\Lambda^2(n-1)m}{m(m+1)^2(m+3)(m+n-1)^2} r^4 + O(r)^6,\end{aligned}\tag{2.10}$$

in terms of one parameter $a(0)$. As we shall see, this parameter is not arbitrary, being uniquely fixed by numerics.

The solitons are asymptotically locally AdS as $r \rightarrow \infty$. For even d , the solution of the Einstein equations admits at large r a power series expansion of the form:

$$\begin{aligned}a(r) &= \frac{r^2}{\ell^2} + \sum_{k=0}^{(d-4)/2} a_k \left(\frac{\ell}{r}\right)^{2k} - M \left(\frac{\ell}{r}\right)^{d-3} + O(1/r^{d-2}), \\ f(r) &= \frac{r^2}{\ell^2} + \sum_{k=0}^{(d-4)/2} f_k \left(\frac{\ell}{r}\right)^{2k} - (d-m-1)M \left(\frac{\ell}{r}\right)^{d-3} + O(1/r^{d-2}),\end{aligned}\tag{2.11}$$

with a_k, f_k are constants depending on the spacetime dimension and the value of m only. Specifically, we find

$$\begin{aligned}a_0 &= \frac{m-1}{d-3}, \quad a_1 = \frac{(m-1)^2(d-m-2)}{(d-2)(d-3)^2(d-5)}, \\ a_2 &= -\frac{(m-1)^3(d-m-2)(38+4m+d(d+2m-21))}{3(d-2)^2(d-3)^3(d-5)(d-7)},\end{aligned}\tag{2.12}$$

and

$$\begin{aligned}f_0 &= \frac{(m-1)(2d-m-4)}{(d-2)(d-3)}, \quad f_1 = \frac{(m-1)^2(d-m-1)(d-m-2)}{(d-2)(d-3)^2(d-5)}, \\ f_2 &= -\frac{(m-1)^3(d-m-2)(d-m-1)(8+d(m-4)-m)}{(d-2)^2(d-3)^3(d-5)(d-7)},\end{aligned}\tag{2.13}$$

their expressions becoming more complicated for higher k , without exhibiting a general pattern.

The corresponding expansion for odd values of the spacetime dimension is more complicated, with $\log(r/\ell)$ terms:

$$a(r) = \frac{r^2}{\ell^2} + \sum_{k=0}^{(d-5)/2} a_k \left(\frac{\ell}{r}\right)^{2k} + \alpha \log\left(\frac{r}{\ell}\right) \left(\frac{\ell}{r}\right)^{d-3} - M \left(\frac{\ell}{r}\right)^{d-3} + O\left(\frac{\log r}{r^{d-1}}\right), \quad (2.14)$$

$$f(r) = \frac{r^2}{\ell^2} + \sum_{k=0}^{(d-5)/2} f_k \left(\frac{\ell}{r}\right)^{2k} + (d-m-1) \alpha \log\left(\frac{r}{\ell}\right) \left(\frac{\ell}{r}\right)^{d-3} - (d-m-1)(M+c_0) \left(\frac{\ell}{r}\right)^{d-3} + O\left(\frac{\log r}{r^{d-1}}\right),$$

with a_k, f_k still given by (2.12), (2.13) and α and c_0 two new constants depending on both d and m . One finds that α can be expressed in a compact form as

$$\alpha = a_{(d-3)/2} \sum_{k>0} (d-2k-1) \delta_{d,2k+1}, \quad (2.15)$$

e.g. $\alpha = -(m-1)^2(m-3)/12$ for $d=5$, $\alpha = (m-1)^3(m-5)(3m-10)/1600$ for $d=7$, and $\alpha = (m-1)^4(m-7)(2m-7)(17m-91)/1778112$ for $d=9$. The constant c_0 is given by

$$c_0 = 0\delta_{5,d} - \frac{3}{3200}\delta_{7,d} - \frac{1475}{42674688}\delta_{9,d} + \dots \quad \text{for } m=2, \quad (2.16)$$

$$c_0 = -\frac{1}{300}\delta_{7,d} - \frac{4}{83349}\delta_{9,d} + \dots \quad \text{for } m=3, \quad c_0 = \frac{129}{175616}\delta_{9,d} - \frac{85}{21233664}\delta_{11,d} + \dots \quad \text{for } m=4.$$

In the above relations (2.11), (2.14), M is an unknown constant which is uniquely determined by the requirement of bulk regularity. The dots denote terms that are subleading orders in r relative to the terms written above (also, no new parameter appears in these subleading terms).

Note that in principle the AdS soliton with $m=1$ can also be written in the gauge used above, $P(r)=r$, such that the range of the radial coordinate becomes $0 \leq r < \infty$. However, the corresponding expression of the functions a and f are quite complicated for this choice. One finds *e.g.*

$$f(r) = \frac{2r^2}{\ell^2} \frac{1 - 4M(\frac{\ell}{r})^4}{1 + \sqrt{1 - 4M(\frac{\ell}{r})^4}}, \quad a(r) = \frac{r^2}{2\ell^2} \left(1 + \sqrt{1 - 4M(\frac{\ell}{r})^4} \right), \quad \text{for } d=5, \quad (2.17)$$

and

$$f(r) = \frac{4}{U(r)} \left(r^2 - \frac{3U(r)}{2\ell^2} \right)^2, \quad a(r) = \frac{U(r)}{\ell^4}, \quad \text{for } d=7, \quad (2.18)$$

where

$$U(r) = \frac{\ell^2}{6} \left(2r^2 + \frac{2^{4/3}r^4}{\left(2r^6 - 27M\ell^6 + 3\sqrt{3}\ell^3\sqrt{M(-4r^6 + 27M\ell^6)} \right)^{1/3}} \right. \\ \left. + 2^{2/3} \left(2r^6 - 27M\ell^6 + 3\sqrt{3}\ell^3\sqrt{M(-4r^6 + 27M\ell^6)} \right)^{1/3} \right).$$

Also, from the point of view of the expansions for $r \rightarrow 0$ and $r \rightarrow \infty$, the $m = 1$ AdS solitons are different. For example, as $r \rightarrow 0$, one finds in that case

$$f(r) = f(0) - \frac{(d-1)(d-4)}{2\ell^2}r^2 + \frac{(d-1)^2(d-2)(d-3)}{16f(0)\ell^4}r^2 + \dots, \quad (2.19)$$

$$a(r) = a(0)\left(1 + \frac{(d-1)}{2f(0)\ell^2}r^2 + \frac{(d-1)^2(d-3)}{16f(0)\ell^4}r^4\right) + \dots, \quad (2.20)$$

instead of (2.10) (with $f(0) = (d-1)|M|^{2/(d-1)}$ and $a(0) = |M|^{2/(d-1)}$), and

$$f(r) = \frac{r^2}{\ell^2} - (d-2)M\left(\frac{\ell}{r}\right)^{d-3} + O(1/r^{d+1}), \quad a(r) = \frac{r^2}{\ell^2} - M\left(\frac{\ell}{r}\right)^{d-3} + O(1/r^{d+1}), \quad (2.21)$$

at infinity (and thus $\log r$ terms are absent in the asymptotics for any d). Note also that for $m = 1$ solutions, the mass parameter M takes only *negative* values.

2.4 A mass computation

The action and mass of the new AdS configurations are computed by using the boundary counterterm prescription [20]. In the usual approach (*e.g.* for the $m = 1$ AdS solitons or the Schwarzschild-AdS black holes) the following boundary counterterm part is added to the action (2.1) [20, 21]:

$$\begin{aligned} I_{\text{ct}}^0 = & \frac{1}{8\pi G} \int d^{d-1}x \sqrt{-\gamma} \left\{ -\frac{d-2}{\ell} - \frac{\ell\Theta(d-4)}{2(d-3)}R - \frac{\ell^3\Theta(d-6)}{2(d-3)^2(d-5)} \left(R_{ab}R^{ab} - \frac{d-1}{4(d-2)}R^2 \right) \right. \\ & + \frac{\ell^5\Theta(d-8)}{(d-3)^3(d-5)(d-7)} \left(\frac{3d-1}{4(d-2)}RR^{ab}R_{ab} - \frac{d^2-1}{16(d-2)^2}R^3 \right. \\ & \left. \left. - 2R^{ab}R^{cd}R_{acbd} - \frac{d-1}{4(d-2)}\nabla_a R \nabla^a R + \nabla^c R^{ab} \nabla_c R_{ab} \right) + \dots \right\}, \end{aligned} \quad (2.22)$$

where R and R^{ab} are the curvature and the Ricci tensor associated with the induced metric γ . The series truncates for any fixed dimension, with new terms entering at every new even value of d , as denoted by the step-function ($\Theta(x) = 1$ provided $x \geq 0$, and vanishes otherwise).

However, given the presence for odd d of $\log(r/\ell)$ terms in the asymptotic expansions of the metric functions (with r the radial coordinate), the counterterms (2.22) regularise the action for even dimensions only. For odd values of d , we have to add the following extra terms to (2.1) [22]:

$$\begin{aligned} I_{\text{ct}}^s = & \frac{1}{8\pi G} \int d^{d-1}x \sqrt{-\gamma} \log\left(\frac{r}{\ell}\right) \left\{ \delta_{d,5} \frac{\ell^3}{8} \left(\frac{1}{3}R^2 - R_{ab}R^{ab} \right) \right. \\ & \left. - \frac{\ell^5}{128} \left(RR^{ab}R_{ab} - \frac{3}{25}R^3 - 2R^{ab}R^{cd}R_{acbd} - \frac{1}{10}R^{ab}\nabla_a \nabla_b R + R^{ab}\square R_{ab} - \frac{1}{10}R\square R \right) \delta_{d,7} + \dots \right\}. \end{aligned}$$

Using these counterterms in odd and even dimensions, one can construct a divergence-free boundary stress tensor from the total action $I = I_0 + I_{\text{ct}}^0 + I_{\text{ct}}^s$ by defining a boundary stress-tensor:

$$T_{ab} = \frac{2}{\sqrt{-\gamma}} \frac{\delta I}{\delta \gamma^{ab}}, \quad (2.23)$$

whose explicit expression for $d \leq 9$ is given in ref. [21]. Thus a conserved charge

$$\mathfrak{Q}_\xi = \oint_\Sigma d^{d-2} S^a \xi^b T_{ab}, \quad (2.24)$$

can be associated with a closed surface Σ (with normal n^a), provided the boundary geometry has an isometry generated by a Killing vector ξ^a [23].

The conserved mass/energy \mathcal{M} is the charge associated with the time translation symmetry, with $\xi = \partial/\partial t$. A straightforward computation leads to the following expressions for the mass of the new AdS solitons:

$$\mathcal{M} = \frac{\ell^{m-1} m \Omega_m V_x}{16\pi G} (M + M_c^{(m,d)}), \quad (2.25)$$

with $\Omega_m = 2\pi^{(m+1)/2}/\Gamma((m+1)/2)$ is the total area of a unit sphere in m -dimensions. M is the constant which enters the large r expansion (2.11), (2.14) of the solutions.

Also, $M_c^{(m,d)}$ are Casimir-like terms which appear for an odd spacetime dimension only,

$$\begin{aligned} M_c^{(m,d)} = & \delta_{m,2} \left(\frac{1}{24} \delta_{d,5} - \frac{1}{3200} \delta_{d,7} - \frac{25}{13333584} \delta_{d,9} + \dots \right) \\ & + \delta_{m,3} \left(-\frac{7}{4800} \delta_{d,7} + \frac{221}{5334336} \delta_{d,9} + \dots \right) + \delta_{m,4} \left(\frac{67}{351232} \delta_{d,9} + \dots \right) + \dots \end{aligned} \quad (2.26)$$

At least formally, one can define a second charge of these solutions, which is the tension associated with translations around some direction x^k . Supposing some periodicity L_k for this coordinate, one finds from (2.24) the tension

$$\mathcal{T}_k = -\frac{\ell^{m-1} m \Omega_m V_x}{16\pi G L_k} (M + M_c^{(m,d)}), \quad (2.27)$$

which is fixed by the mass \mathcal{M} of the solutions⁶. For an infinite L_k , the value of V_x also diverges and one has to work with mass and tension densities (however, for the numerical calculations in this work, the values of V_x and L_k are not relevant).

Also, it is straightforward to show that, in the absence of an event horizon, the action of a soliton is given by $I = \beta \mathcal{M}$, with β the (arbitrary) periodicity of the Euclidean time coordinate.

2.5 Numerical solutions

The numerical evaluation of these solutions requires some care, since the constant M appears as subleading term in the Fefferman-Graham expansion (2.11), (2.14).

The solutions of the equations (2.4) were found by using two different methods. First, by employing a standard ordinary differential equation solver, we evaluate the initial conditions (2.10) at $r = 10^{-6}$ for global tolerance 10^{-12} , adjusting the shooting parameter $a(0)$ and

⁶This is consistent with the interchange symmetry of this type of configurations, see *e.g.* the discussion for $m = 1$ in [24].

	d	M	$a(0)$
$m = 2$	5	-0.1075	0.6518
	6	-0.0205	0.4703
	7	-0.0096	0.3617
	8	-0.0019	0.2907
	9	-0.0006	0.2412
	10	-0.0001	0.2050
$m = 3$	6	0.0801	0.7440
	7	0.0262	0.5861
	8	0.0083	0.4801
	9	0.0026	0.4071
	10	0.0007	0.3485
$m = 4$	7	0.0439	0.7970
	8	0.0106	0.6590
	9	0.0032	0.5597
	10	0.0011	0.4851

Table 1. The values of the asymptotic mass parameter M and of the metric function $a(r)$ at the origin are shown for $m = 2, 3, 4$ solitons and several values of the spacetime dimension d with four digits accuracy.

integrating towards $r \rightarrow \infty$. In practice, the integration stops for some r_{max} where the asymptotic limit (2.11), (2.14) is reached with reasonable accuracy (typically we have taken $r_{max} \sim 20\ell$). In a different approach, we have integrated the non linear ordinary differential equation (2.8) with the boundary condition $A(0) = 0$, by using a standard solver [25]. This solver involves a Newton-Raphson method for boundary-value ordinary differential equations, equipped with an adaptive mesh selection procedure. Typical mesh sizes include $10^2 - 10^3$ points; also we have used in this case a compactified radial coordinate $x = r/(1 + r)$, such that the domain of integration is $0 \leq x \leq 1$.

AdS solutions with the asymptotics (2.11), (2.14) were found for $m = 2, 3, 4, 5$ and values of $d \leq 10$. Therefore we expect such configurations to exist for any $(n, m) \geq 2$. Our numerical calculations indicate that, given (d, m, Λ) , solutions with the right asymptotics exist only for a single value of $a(0)$. This uniquely fixes also the value of the asymptotic mass-parameter M which enters the large r form of the solutions (2.11), (2.14). The values of M and $a(0)$ are shown in Table 1 for $m = 2, 3, 4$ solitons⁷ with $d \leq 10$ (note that only the first four digits are given there). One can see that all considered solutions with $m = 2$ have a negative

⁷The appearance of these values of $a(0), M$, together with the complicated form of the $m = 1$ configurations (2.17), (2.18) suggests that analytical solutions, if they exist, should be sought for another metric ansatz than (2.3).

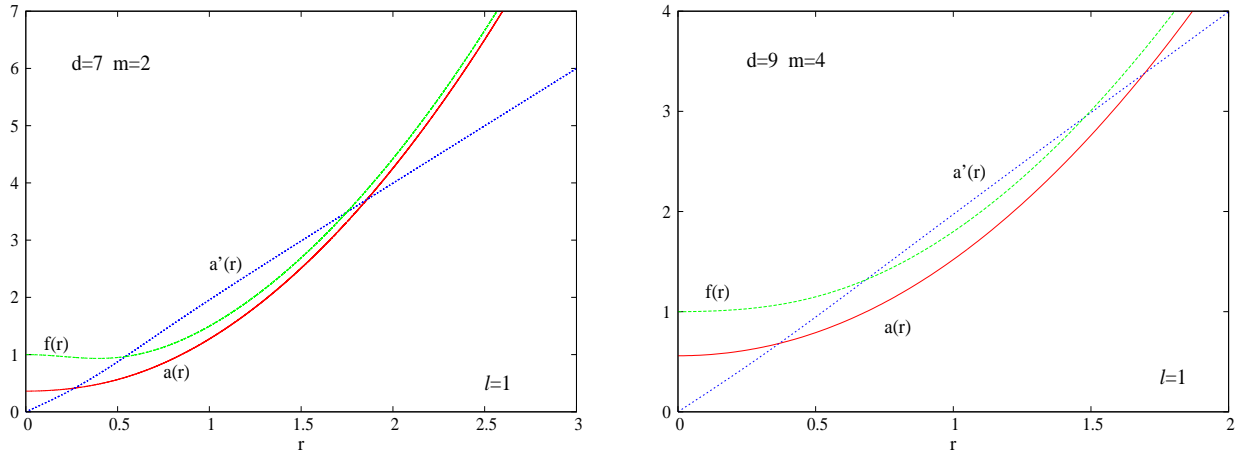


Figure 1: The profiles of the metric functions $a(r)$, $f(r)$ and of the derivative of $a(r)$ are shown for typical soliton solutions with $\ell = 1$.

mass-parameter M (we recall that the $m = 1$ solitons also have $M < 0$, which is an arbitrary parameter). However, this is not the case for the considered $m = 3, 4$ configurations. Also, intriguingly enough, for both $m = 2$ and $m = 3$, the values of $\log |M|$ have almost a linear dependence on d .

The results in Table 1 are found for $\ell = 1$. The solutions for any other negative values of the cosmological constant are easily found by using a suitable rescaling of the $\ell = 1$ configurations. Indeed, the Einstein equations (4.3) are left invariant by the transformation

$$r \rightarrow \bar{r} = \lambda r, \quad \ell \rightarrow \bar{\ell} = \lambda \ell. \quad (2.28)$$

and thus the mass of a soliton scales as $\mathcal{M} \rightarrow \bar{\mathcal{M}} = \lambda^{m-1} \mathcal{M}$.

For all configurations we have studied, $a(r)$ and $f(r)$ are smooth functions interpolating between the corresponding values at $r = 0$ and the asymptotic values at infinity. We find that $a(r)$ is increasing monotonically while $f(r)$ may possess a local extremum around $r \sim \ell$. For large r , the functions are proportional to r^2/ℓ^2 , indicating that the solutions are asymptotically locally AdS as $r \rightarrow \infty$. The profiles of the metric functions of typical $m = 2, 4$ soliton solutions are presented in Figure 1.

3. Application: brane world models with compact extra-dimensions

The proposal of Randall and Sundrum (RS) to localize gravity in the vicinity of a brane with nonvanishing tension in an AdS bulk [12] has attracted enormous attention in the last decade. The RS construction consists in taking two copies of a part of the five dimensional AdS metric in Poincaré coordinates and gluing them together along a boundary which is interpreted as a three brane. In this approach, the four-dimensional gravity naturally arises at long distances on the brane and the solution to the Einstein equations results in a single

graviton zero mode (which is a consequence of the unbroken $4d$ Poincaré invariance) and a continuum of Kaluza-Klein modes.

Based on the globally soliton solutions in Section 2, we propose in what follows a generalization of the RS model, the brane possessing in this case an extra part which is a round sphere S^m . The existence of $m \geq 1$ compact extradimensions is also a feature of Kaluza-Klein models. However, the situation is quite different for the RS inspired scenario in this work. Also, although the case of interest here is $n = 4$, for the sake of generality, we shall not fix the dimensionality of the flat part of the metric.

Note that the model with a general m has some similarities with the $m = 1$ six dimensional warped brane worlds considered in various contexts by several authors [26]. However, most the results in this Section apply also for $m = 1$, in particular the issue of a black hole on the brane.

3.1 The junction conditions

In order to apply the RS approach to the type (2.3) of metrics, we start by defining a new radial coordinate

$$z = \int \frac{dr}{\sqrt{f(r)}} \quad (3.1)$$

such that $z = r - f_2 r^3/6 + \dots$ as $r \rightarrow 0$ (with $f_2 = \frac{4\Lambda^2(n-1)m}{m(m+1)^2(m+3)(m+n-1)^2}$). For large r/ℓ , one finds the usual relation $r \simeq e^{z/\ell}$. The metric (2.3) is now written in a holographic-like form⁸

$$ds^2 = dz^2 + a(z)d\Sigma_n^2 + r^2(z)d\omega_m^2. \quad (3.2)$$

Now we consider the brane to be located at a given distance z_0 from the origin $z = 0$. The brane construction implies that we must keep the region $0 \leq z \leq z_0$ of the bulk. Then, by a similar orbifold procedure as in the RS model, we replace the part of the spacetime outside the brane ($z > z_0$) with a copy of the inner part, ending up with a finite range for the new 'radial' variable⁹ z .

The induced metric on the brane located at $z = z_0 > 0$ corresponds to a direct product $M_n \times S^m$, but with unequal radii R_1 and R_2 for M_n and S^m respectively,

$$d\sigma^2 = R_1^2 d\Sigma_n^2 + R_2^2 d\omega_m^2, \quad (3.3)$$

with $R_1 = \sqrt{a(z_0)}$, $R_2 = r(z_0)$. Note that only the ratio R_2/R_1 is relevant here, since one can always set $R_1 = 1$ by using a suitable rescaling. Therefore the value of R_2 can be arbitrarily small, for a position of the brane close to the bulk origin, $z = 0$.

⁸In principle, the $m > 1$ AdS solitons can be directly constructed within the parametrization (3.2). However, the numerics becomes more difficult in this case and we could not extract the asymptotic coefficient M with enough accuracy.

⁹Formally, one may define a new variable $\bar{z} = z_0 - z$ such that the modulus of the coordinate \bar{z} runs from the position of the brane at $\bar{z} = 0$ and the origin of the spacetime $|\bar{z}| = z_0$.

The geometry of the RS model is also given by (3.2), (3.3) with $m = 0$ and $a(z) = e^{-2|z|/\ell}$. The range of the 'radial' coordinate here is unbounded, $\ell \leq z < \infty$, with $z \rightarrow \infty$ corresponding to a (bulk) horizon. Thus, from some point of view, the compact extradimensions on the brane affect also the bulk geometry and convert the AdS horizon into a regular origin.

Typically, a brane geometry is supported by some matter fields confined on the brane. These matter fields have an energy-momentum tensor T_{ij} which enters the Israel junction conditions on the brane

$$K_{ij} - Kh_{ij} = \frac{\kappa^2}{2}(-\sigma h_{ij} + T_{ij}), \quad (3.4)$$

where $\kappa^2 = 8\pi G$, K_{ij} is the extrinsic curvature tensor and σ is the brane tension (which can also be thought as a kind of matter distribution).

The Israel junction conditions applied to a brane world (3.2) with compact dimensions lead to the following set of equations

$$m \frac{\dot{r}(z_0)}{r_0} + (n-1) \frac{\dot{a}(z_0)}{2a_0} = \frac{\kappa^2}{2}(-T_x^x + \sigma), \quad (m-1) \frac{\dot{r}(z_0)}{r_0} + n \frac{\dot{a}(z_0)}{2a_0} = \frac{\kappa^2}{2}(-T_\phi^\phi + \sigma), \quad (3.5)$$

where T_x^x and T_ϕ^ϕ are the relevant nonvanishing components of the energy-momentum tensor of the matter fields on the brane, with x a direction on M_n and ϕ an angle on the m -dimensional sphere (thus $T_i^j = T_x^x \delta_{x^i}^{x^j} + T_\phi^\phi \delta_{\varphi^i}^{\varphi^j}$). Also, $a_0 = a(z_0)$, $r_0 = r(z_0)$ and a dot denotes a derivative with respect to z (we recall $d/dz = \sqrt{f}d/dr$).

3.2 Matter sources for the brane world

From (3.5), one can see that the energy-momentum tensor of the matter fields on the brane is anisotropic, since

$$\frac{\kappa^2}{2}(T_\phi^\phi - T_x^x) = \frac{\dot{r}(z_0)}{r_0} - \frac{\dot{a}(z_0)}{2a_0} > 0, \quad (3.6)$$

for any choice of z_0 (this is implied by the numerical results in Section 2). Therefore one cannot interpret T_i^j as being due to a pure tension.

However, one can imagine different mechanisms which lead to an energy-momentum tensor consistent with (3.5). Perhaps the most natural possibility is to take some matter fields effectively living on the S^m part of the brane world metric. A simple choice here is to consider a multiplet of $(m+1)$ scalar fields Φ^a , with a lagrangean density (the index i runs over all coordinates on the brane)

$$L = -\frac{1}{2}\partial_i\Phi^a\partial^i\Phi^a - V(\Phi), \quad (3.7)$$

with some potential $V(\Phi)$. Then we consider a hedgehog configuration for these scalars with

$$\Phi^a = \phi(r_0)\hat{u}_a \quad (3.8)$$

with \hat{u}_a a unit vector depending on the coordinates on S^m only¹⁰ and $\phi(r_0)$ is the amplitude of the scalars which is constant, $\phi(r_0) = \eta$.

This leads to the simple relations

$$\kappa^2 \eta^2 = 2r_0^2 \left(\frac{\dot{r}(z_0)}{r_0} - \frac{\dot{a}(z_0)}{2a_0} \right), \quad \kappa^2 \sigma = \frac{m\dot{r}(z_0)}{r_0} + (m + 2(n - 1)) \frac{\dot{a}(z_0)}{2a_0} - \kappa^2 V(\eta), \quad (3.9)$$

for the value of the scalar field and brane tension. Based on the expansion (2.10), one can write an approximate form for η^2 and σ for small values of z_0 (*i.e.* $z \rightarrow 0$)

$$\kappa^2 \sigma = \frac{m}{z_0} - \kappa^2 V(\eta) + \frac{\Lambda}{12} (m - 16) z_0 + O(z_0^3), \quad \kappa^2 \eta^2 = 2z_0 + O(z_0^3), \quad (3.10)$$

and, from (2.11), (2.14), for $z_0 \rightarrow \infty$

$$\kappa^2 \sigma = \frac{2(d - 2)}{\ell} - \kappa^2 V(\eta) + \dots, \quad \kappa^2 \eta^2 = \frac{2(m - 1)\ell}{(d - 3)} + \dots \quad (3.11)$$

The scalar field equation

$$\frac{m\eta}{r_0^2} = \frac{\partial V}{\partial \phi} \Big|_{\eta}, \quad (3.12)$$

provides an extra-constraint which implies that η cannot be an extremum of the potential. Nevertheless, a simple quadratic scalar potential, $V = \mu^2 \phi^2$, is compatible with (3.12).

However, the choice of the matter fields on the brane proposed above is not unique. For example, a brane world with a compact S^2 can be supported by a magnetic monopole. In this case one considers a $U(1)$ field with

$$L = -\frac{1}{4} F_{ij} F^{ij}, \quad (3.13)$$

where $F = dA$. For an abelian magnetic monopole, the only component of the $U(1)$ field potential is $A = Q_M \cos \varphi_1 d\varphi_2$ (with $d\omega_2^2 = d\varphi_1^2 + \sin^2 \varphi_1 d\varphi_2^2$). The Israel junction conditions lead to the following equations

$$\kappa^2 \sigma = \frac{(2d - 7)\dot{a}(z_0)}{2a_0} + \frac{3\dot{r}(z_0)}{r_0}, \quad \frac{\kappa^2 Q_M^2}{2r_0^4} = -\frac{\dot{a}(z_0)}{2a_0} + \frac{\dot{r}(z_0)}{r_0}, \quad (3.14)$$

which fix the brane tension and the value of the magnetic charge Q_M as a function of the position of the brane and the bulk geometry.

The $m > 2$ brane worlds are supported by higher dimensional nonabelian generalizations of the Dirac monopole. The Lagrangean density in this case is

$$L = -\frac{1}{4g^2} \text{Tr}\{F_{ij} F^{ij}\}, \quad (3.15)$$

¹⁰Specifically, if one takes the metric on S^m as $d\omega_m^2 = d\varphi_1^2 + \sin^2 \varphi_1 (d\varphi_2^2 + \dots + \sin^2 \varphi_{m-2} (d\varphi_{m-1}^2 + \sin^2 \varphi_{m-1}^2 d\varphi_m^2) \dots)$, one writes $\hat{u}_1 = \cos \varphi_1$, $\hat{u}_2 = \sin \varphi_1 \cos \varphi_2, \dots, \hat{u}_{m+1} = \sin \varphi_1 \sin \varphi_2 \dots \sin \varphi_m$.

$F_{kj} = \partial_k A_j - \partial_j A_k - i[A_k, A_j]$ being the gauge field strength tensor and g the gauge coupling constant.

In what follows we shall use the notations and conventions of [27]. Adopting the criterion of employing chiral representations, both for *even* and for *odd* m , it is convenient to choose the gauge group to be $SO(\bar{m})$. We shall therefore denote our representation matrices by $SO_{\pm}(\bar{m})$, where $\bar{m} = m + 2$ and $\bar{m} = m + 1$ for *even* and *odd* m respectively. In this unified notation (for odd and even m), the spherically symmetric Ansatz for the $SO_{\pm}(\bar{m})$ -valued gauge fields then reads [27]

$$A_i = \left(\frac{w(r_0) - 1}{r_0} \right) \Sigma_{ij}^{(\pm)} \hat{u}_j, \quad \text{with} \quad \Sigma_{ij}^{(\pm)} = -\frac{1}{4} \left(\frac{1 \pm \Gamma_{\bar{m}+1}}{2} \right) [\Gamma_i, \Gamma_j], \quad (3.16)$$

where the Γ 's denote the \bar{m} -dimensional gamma matrices and $i, j = 1, 2, \dots, m + 1$ for both cases. Also, \hat{u}_j is a unit vector depending on the coordinates on S^m only (see footnote 10).

One can easily verify that $w(r_0) = 0$ is a solution of the Yang-Mills equations. Thus the nonabelian solutions are such that the field strength has components on the m -sphere only (and thus they are essentially different from the higher dimensional nonabelian solutions reviewed in [28]), being akin to the Yang monopoles in [29], [30].

A straightforward computation based on (3.4) leads to the following equations

$$\kappa^2 \sigma = \frac{3m}{2} \frac{\dot{r}(z_0)}{r_0} + (4d - 3m - 8) \frac{\dot{a}(z_0)}{4a_0}, \quad \kappa^2 \frac{n_{\bar{m}}}{4g^2} = \frac{r_0^4}{2(m-1)} \left(\frac{\dot{r}(z_0)}{r_0} - \frac{\dot{a}(z_0)}{2a_0} \right), \quad (3.17)$$

(with $n_{\bar{m}} = \text{Tr } \mathbb{I}$, where the dimensionality of the unit matrix is determined by the chiral representations appearing in (3.16)) which fix the brane tension and the gauge coupling constant as a function of the brane's position.

For $m = 4$, one can take instead a BPST nonabelian instanton [31] to support the junction conditions (3.4). The matter lagrangean in this case is still given by (3.15). The gauge group is $SU(2)$ with a gauge field potential

$$A^a = (w(\varphi_1) + 1) \theta^a. \quad (3.18)$$

In this relation, θ^a are the left-invariant forms on S^3 , with $d\omega_3^2 = \delta_{a,b} \theta^a \theta^b$, the metric on the four sphere being written as $d\omega_4^2 = d\varphi_1^2 + \sin^2 \varphi_1 d\omega_3^2$. This gauge field is a self-dual solution of the Yang-Mills equations on the brane for $w(r) = \cos \varphi_1$, and thus has $T_\phi^\phi = 0$. However, the components of the energy momentum tensor along the flat directions T_x^x are nonvanishing.

Then the junction conditions (3.4) lead to the following relations for the gauge coupling constant and brane tension as a function of the position of the brane:

$$\frac{\kappa^2 \sigma}{2} = \frac{(d-5)}{2} \frac{\dot{a}(z_0)}{a_0} + \frac{3\dot{r}(z_0)}{r_0}, \quad \frac{\kappa^2}{8g^2 r_0^4} = \frac{\dot{r}(z_0)}{r_0} - \frac{\dot{a}(z_0)}{2a_0}. \quad (3.19)$$

By using the generalizations of the BPST instanton in [32], one can find a similar solution for any dimension $m = 4p$ of the sphere.

Other choices of possible sources to support the junction conditions (3.4) seems to be possible¹¹. It would be interesting to explore the validity for $m > 1$ of a number of proposals employed in models with a single extra-direction [26].

3.3 Black hole on a brane with compact extra-dimensions

A curious problem of the RS model consists in the absence of a satisfactory solution for a black hole on the brane¹², despite of a large amount of work in this direction. The simplest proposal consists in replacing the Minkowski metric on the brane by a Schwarzschild black hole [38]. This results in the bulk geometry

$$ds^2 = dz^2 + e^{-2|z|/\ell} \left(\frac{d\rho^2}{1 - (\frac{\rho_0}{\rho})^{d-4}} + \rho^2 d\omega_{d-3}^2 - (1 - (\frac{\rho_0}{\rho})^{d-4}) dt^2 \right). \quad (3.20)$$

One problem with this proposal is that the $\rho = 0$ singularity extends all the way out to the AdS horizon and at this surface the solution becomes nakedly singular¹³ [38]. Moreover, this solution suffers from a classical Gregory-Laflamme instability [39].

Here we argue that the naked singularity is absent for models with compact extra-dimensions on the brane. The simplest black hole solution in this case is found by taking again $d\Sigma_n^2$ to represent the Schwarzschild black hole in $d - m - 1$ dimensions

$$ds^2 = dz^2 + a(z) \left(\frac{d\rho^2}{1 - (\frac{\rho_0}{\rho})^{n-3}} + \rho^2 d\omega_{n-2}^2 - (1 - (\frac{\rho_0}{\rho})^{n-3}) dt^2 \right) + r^2(z) d\omega_m^2. \quad (3.21)$$

Although similar to the RS case, the $\rho = 0$ singularity extends out in the bulk all the way to $z = 0$, this time the Kretschmann scalar is finite for any value of z and $\rho \neq 0$. For example, based on the expansion (2.10), one finds as $z \rightarrow 0$,

$$R_{ijkl}R^{ijkl} = f_0(\rho) + f_2(\rho)z^2 + O(z^4), \quad (3.22)$$

where $f_0(\rho)$, $f_2(\rho)$ are functions of ρ depending on m, d and diverging as $\rho \rightarrow 0$ only. One finds *e.g.* $f_0(\rho) = 68\Lambda^2/75 + 12\rho_0^2/(\rho^6/a(0)^2)$, $f_2(\rho) = 64\Lambda^3/225 + 16\Lambda\rho_0^2/(5a(0)^2\rho^6)$ for $m = 2, d = 7$ and $f_0(\rho) = 66\Lambda^2/24 + 12\rho_0^2/(\rho^6/a(0)^2)$, $f_2(\rho) = 96\Lambda^3/6125 + 48\Lambda\rho_0^2/(35a(0)^2\rho^6)$ for $m = 4, d = 9$. The different behaviour as compared to the RS model originates in the different properties of the bulk metrics. In the RS model, the AdS origin corresponds to a

¹¹The possibility to employ the stress tensor of a quantum field in a $M_n \times S^n$ geometry is especially worth investigating. The results in [33] show that such metrics appear as solutions of the Einstein gravity plus free massless fields equations in $4 + m$ dimensions. The energy momentum tensor responsible for the curvature is produced by the quantum fluctuations in the matter fields.

¹²Note that there are a number of theoretical arguments against the existence of *static* black holes on the brane in the RS model, mainly based on a version of the conjectured AdS/CFT correspondence [34], [35]. This seems to be confirmed by the recent numerical results in [36], which indicate that the static black holes on the brane in [37] are essentially numerical artifacts.

¹³This can be seen by computing the Kretschmann scalar, which diverges as $\rho_0^{2(d-4)} e^{4z/\ell} / \rho^{2(d-2)}$ as $z \rightarrow \infty$.

horizon which is infinitely far from the brane (although it can be reached by an observer in finite proper time). For the AdS solitons in this work, the horizon is absent and the bulk origin is at finite proper distance from the brane.

It would be interesting to study the classical stability of these solutions. We expect that the black holes with large enough values of ρ_0 as compared to z_0 do not possess a Gregory-Laflamme instability.

Another important problem to consider in future work is the spectrum of linearized gravity fluctuation around the metric (3.2). In the absence of an exact solution (except for $m = 1$), this would be a more difficult problem than in the RS case. However, we expect that the standard $1/r^{n-3}$ gravitational potential is recovered on M_n for distances much larger than the radius of the sphere S^m .

4. New black hole solutions

The black holes with a cosmological constant $\Lambda < 0$ are of special interest in the AdS/CFT context, since they offer the possibility of studying the nonperturbative structure of some CFTs. For example, the Schwarzschild-AdS₅ Hawking-Page phase transition [40] was interpreted as a thermal phase transition from a confining to a deconfining phase in the dual $d = 4$, $\mathcal{N} = 4$ super Yang-Mills theory [41].

Similar to the $\Lambda = 0$ limit, the Schwarzschild-AdS black hole solution in d dimensions has an event horizon of topology S^{d-2} , which matches the S^{d-2} topology of the spacelike infinity. However, in the presence of a negative cosmological constant, the horizon of black hole solutions admits a much larger variety of geometries and topologies than in the asymptotically flat case. For example, in what follows we present arguments that the solitons discussed in Section 2 have black hole generalizations with an event horizon topology $R^{n-1} \times S^m$. These solutions resemble the known Schwarzschild black branes in Kaluza-Klein theory, since $n - 1$ flat codimensions are present in both cases. However, their asymptotic structure is very different, as well as their thermodynamical properties.

4.1 The equations and asymptotics

In the simplest case, the black hole solutions are constructed within a metric ansatz generalizing (2.3)

$$ds^2 = \frac{dr^2}{f(r)} + r^2 d\omega_m^2 + a(r) \sum_{i,j=1}^{n-1} \delta_{i,j} dx^i dx^j - b(r) dt^2. \quad (4.1)$$

The range of the radial coordinate is restricted here to $r \geq r_h$ with $b(r_h) = f(r_h) = 0$, while $a(r_h) > 0$. Thus $r = r_h > 0$ corresponds to an event horizon.

The Einstein equations with a negative cosmological constant imply that the metric functions $a(r)$, $b(r)$ and $f(r)$ are solutions of the following equations:

$$\begin{aligned}
a'' + (d-m-5)\frac{a'^2}{4a} + \frac{ma'}{r} + \frac{maf'}{(d-m-2)rf} + \frac{a'f'}{2f} + \frac{m(m-1)a}{(d-m-2)r^2}\left(1 - \frac{1}{f}\right) + \frac{2\Lambda a}{(d-m-2)f} &= 0, \\
\frac{b'}{b} + \frac{2m}{r} + \left((d-m-2)(d-m-3)\frac{r^2 a'^2}{a^2} - \frac{4m(m-a)}{f} + \frac{8\Lambda r^2}{f} - 4m(m+1) \right) & \\
\times \left(2r(2m + (d-m-2)\frac{ra'}{a}) \right)^{-1} &= 0, \\
f' + f \left((d-m-2)\frac{a'}{a} + \frac{b'}{b} \right) + \frac{4\Lambda r}{d-2} + \frac{2(m-1)}{r}(f-1) &= 0.
\end{aligned} \tag{4.2}$$

Unfortunately, the solutions of the above equations are known analytically only in special cases. For $m = 0, 1$ one finds

$$f(r) = b(r) = \frac{r^2}{\ell^2} - \left(\frac{r_0}{r}\right)^{d-3}, \quad a(r) = r^2, \tag{4.3}$$

which corresponds to the known topological black hole with a Ricci flat horizon (with a compact direction for $m = 1$), whose properties are reviewed *e.g.* in [42].

For $\Lambda = 0$, one finds instead the black brane solution in [43], with

$$ds^2 = \frac{dr^2}{1 - \left(\frac{r_h}{r}\right)^{d-n-2}} + r^2 d\omega_m^2 + \sum_{i,j=1}^{n-1} \delta_{i,j} dx^i dx^j - \left(1 - \left(\frac{r_h}{r}\right)^{d-n-2}\right) dt^2, \tag{4.4}$$

which is just the Schwarzschild black hole in $m + 2$ dimensions uplifted to d -dimensions.

This suggests to view the black holes discussed in this paper as natural AdS generalizations¹⁴ of the solutions (4.4), *i.e.* as lower dimensional Schwarzschild-AdS black holes uplifted to d -dimensions. A value $\Lambda < 0$ in the bulk leads to a nontrivial $a(r)$, to a product $g_{tt}g_{rr} \neq -1$ and also to a different asymptotic structure as compared to (4.4).

Unfortunately, there is no prescription to uplift a lower dimensional solution to higher dimensions in the presence of a cosmological constant. However, the AdS counterparts of the black $(n-1)$ -branes (4.4) can be studied by using similar methods to those employed in the soliton case.

We assume that close to r_h the metric functions can be expanded¹⁵ into a Taylor series

¹⁴The new AdS solitons in Section 2 can also be interpreted as the AdS_{m+1} regular solution with $ds^2 = \frac{dr^2}{r^2/\ell^2 + 1} + r^2 d\omega_m^2$ uplifted to d -dimensions (thus with n flat extra-directions). This may explain why the solitons could be found only for a single value of the relevant parameters $(a(0), M)$.

¹⁵Note that the expansion (4.5) would not hold for extremal solutions. However, we could not find any indication for the emergence of such configurations. Nevertheless, extremal solutions are likely to exist when adding an extra global charge to the system.

in $r - r_h$, the first terms there being:

$$\begin{aligned}
a(r) &= a_h \left(1 + \frac{2}{r_h} \frac{(d-1) \frac{r_h^2}{\ell^2}}{(d-1) \frac{r_h^2}{\ell^2} + m - 1} (r - r_h) \right) + O(r - r_h)^2, \\
f(r) &= \left((d-1) \frac{r_h^2}{\ell^2} + m - 1 \right) \frac{1}{r_h} (r - r_h) - \left(m(m-1) + (d-1)(d-4) \frac{r_h^2}{\ell^2} \right) \frac{1}{2r_h^2} (r - r_h)^2 + O(r - r_h)^3, \\
b(r) &= b_1 \left((r - r_h) - \frac{m(m-1) + (d-1)(d-4) \frac{r_h^2}{\ell^2}}{(d-1) \frac{r_h^2}{\ell^2} + m - 1} \frac{1}{2r_h} (r - r_h)^2 \right) + O(r - r_h)^3,
\end{aligned} \tag{4.5}$$

in terms of two parameters a_h, b_1 .

Similar to the solitonic limit, the large r Fefferman-Graham expansion of the metric functions is different for odd and even dimensions. For black holes, the asymptotics is written in terms of two constants c_t and c_z which appear as subleading terms in the metric functions. For even d , one finds

$$\begin{aligned}
a(r) &= \frac{r^2}{\ell^2} + \sum_{k=0}^{(d-4)/2} a_k \left(\frac{\ell}{r} \right)^{2k} + c_z \left(\frac{\ell}{r} \right)^{d-3} + O(1/r^{d-2}), \\
b(r) &= \frac{r^2}{\ell^2} + \sum_{k=0}^{(d-4)/2} a_k \left(\frac{\ell}{r} \right)^{2k} + c_t \left(\frac{\ell}{r} \right)^{d-3} + O(1/r^{d-2}), \\
f(r) &= \frac{r^2}{\ell^2} + \sum_{k=0}^{(d-4)/2} f_k \left(\frac{\ell}{r} \right)^{2k} + (c_t + (d-m-2)c_z) \left(\frac{\ell}{r} \right)^{d-3} + O(1/r^{d-2}).
\end{aligned} \tag{4.6}$$

The corresponding expansion for odd values of the spacetime dimension is given by:

$$\begin{aligned}
a(r) &= \frac{r^2}{\ell^2} + \sum_{k=0}^{(d-5)/2} a_k \left(\frac{\ell}{r} \right)^{2k} + \alpha \log\left(\frac{r}{\ell}\right) \left(\frac{\ell}{r} \right)^{d-3} + c_z \left(\frac{\ell}{r} \right)^{d-3} + O\left(\frac{\log r}{r^{d-1}}\right), \\
b(r) &= \frac{r^2}{\ell^2} + \sum_{k=0}^{(d-5)/2} a_k \left(\frac{\ell}{r} \right)^{2k} + \alpha \log\left(\frac{r}{\ell}\right) \left(\frac{\ell}{r} \right)^{d-3} + c_t \left(\frac{\ell}{r} \right)^{d-3} + O\left(\frac{\log r}{r^{d-1}}\right), \\
f(r) &= \frac{r^2}{\ell^2} + \sum_{k=0}^{(d-5)/2} f_k \left(\frac{\ell}{r} \right)^{2k} + (d-m-1) \alpha \log\left(\frac{r}{\ell}\right) \left(\frac{\ell}{r} \right)^{d-3} + (c_t + (d-m-2)c_z) \left(\frac{\ell}{r} \right)^{d-3} + O\left(\frac{\log r}{r^{d-1}}\right),
\end{aligned} \tag{4.7}$$

with α and c_0 given by (2.15) and (2.16), respectively. Also, the expression of the constants a_k, f_k in the above relations is similar to that found for the regular solutions.

For both even and odd dimensions one finds the asymptotic expression of the Riemann tensor $R_{\mu\nu}^{\lambda\sigma} = -(\delta_\mu^\lambda \delta_\nu^\sigma - \delta_\mu^\sigma \delta_\nu^\lambda)/\ell^2 + \dots$, which shows that these solutions are locally asymptotically AdS. Note also that the soliton solutions discussed in Section 2 are recovered as $r_h \rightarrow 0$, in which case $c_t = c_z = -M$.

4.2 The mass computation and a Smarr law

The global charges of the black holes are computed by using the same counterterm approach as in the globally regular case. The computation of the boundary stress tensor T_{ab} is straightforward and we find the following expressions for the mass and tension of a black hole:

$$\mathcal{M} = \frac{\Omega_m V_x \ell^{m-1}}{16\pi G} \left[(d-m-2)c_z - (d-2)c_t \right] + \frac{\Omega_m V_x m \ell^{m-1}}{16\pi G} M_c^{(m,d)}, \quad (4.8)$$

$$\mathcal{T}_k = \frac{\Omega_m V_x \ell^{m-1}}{16\pi G L_k} \left[(m+1)c_z - c_t \right] - \frac{\Omega_m V_x m \ell^{m-1}}{16\pi G L_k} M_c^{(m,d)}, \quad (4.9)$$

with $M_c^{(m,d)}$ a Casimir term given by (2.26) (note that the mass and tension are independent quantities in this case).

We note that the considered Lorentzian solutions extremize also the Euclidean action as the analytical continuation $t \rightarrow i\tau$ has no effects at the level of the equations of motion. The Hawking temperature of these solutions is computed by demanding regularity of the Euclideanized manifold as $r \rightarrow r_h$

$$T_H = \frac{1}{4\pi} \sqrt{\left((d-1) \frac{r_h^2}{\ell^2} + m-1 \right) \frac{b_1}{r_h}}. \quad (4.10)$$

Thus we can proceed further by formulating gravitational thermodynamics via the Euclidean path integral [44]

$$Z = \int D[g] e^{-I[g]} \simeq e^{-I},$$

where one integrates over all metrics and matter fields between some given initial and final Euclidean hypersurfaces, taking τ to have a period $\beta = 1/T_H$. Semiclassically the result is given by the classical action evaluated on the equations of motion, and yields to this order an expression for the entropy

$$S = \beta M - I, \quad (4.11)$$

upon application of the quantum statistical relation to the partition function.

To evaluate the solutions' action, we integrate the Killing identity $\nabla^\mu \nabla_\nu K_\mu = R_{\nu\mu} K^\mu$, for the Killing vector $K^\mu = \delta_t^\mu$, together with the Einstein equation $R_t^t = (R - 2\Lambda)/2$. Thus, we isolate the bulk action contribution at infinity and at $r = r_h$. The divergent contributions given by the surface integral term at infinity (plus the Gibbons-Hawking term) are also canceled by $I_{ct}^0 + I_{ct}^s$ and, together with (4.11), we find as expected $S = A_H/4G$, where

$$A_H = \Omega_m V_x r_h^m a_h^{(d-m-2)/2} \quad (4.12)$$

is the event horizon area.

The same approach applied for a Killing vector $K^\mu = \delta_{x^k}^\mu$ yields the result:

$$I = -\beta \mathcal{T}_k L_k. \quad (4.13)$$

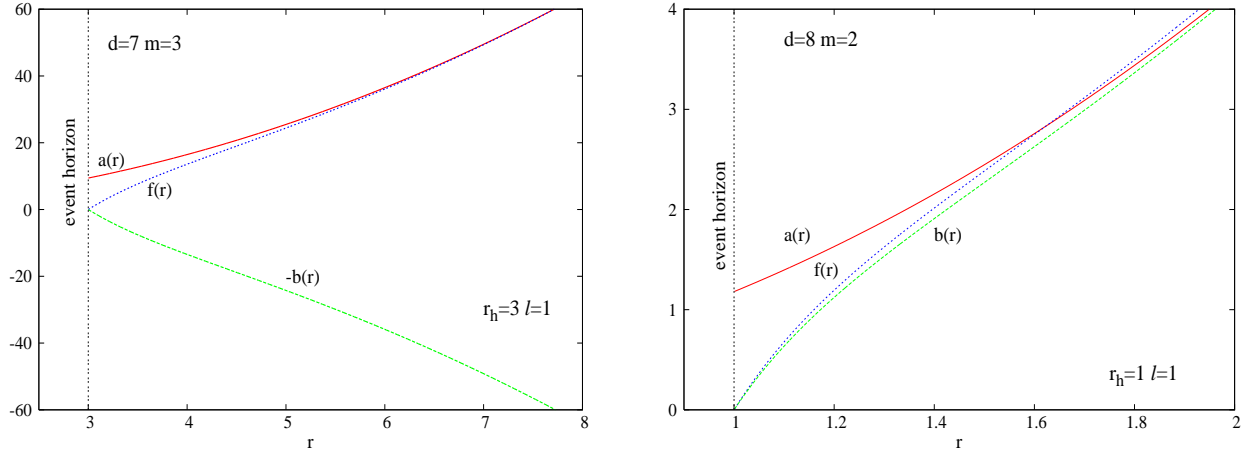


Figure 2: The profiles of the metric functions $a(r)$, $f(r)$ and $f(r)$ are shown for typical black hole solutions.

The relations (4.11) and (4.13) lead to a simple Smarr-type formula, relating quantities defined at infinity to quantities defined at the event horizon:

$$M + \mathcal{T}_k L_k = T_H S . \quad (4.14)$$

This relation also provides an useful check of the numerical accuracy (note that in all numerical data we have set $L_k = V_x = 1$).

4.3 The properties of the solutions

We use the series expansion (4.5) to fix the initial data at $r = r_h + \epsilon$, with $\epsilon = 10^{-6}$. The system (4.3)-(4.2) is then integrated by using a standard ordinary differential equation solver and adjusting for fixed shooting parameters. The integration stops when the asymptotic limit (4.6), (4.7) is reached with sufficient accuracy. Given (m, d, Λ, r_h) , solutions with the right asymptotics are found for one set of the shooting parameters (a_h, b_1) only.

The results we present here are obtained for $\ell = 1$. However, similar to the soliton case, the solutions for any other value of the cosmological constant are found by using a suitable rescaling of these configurations. The effects of the transformation (2.28) on the black hole solutions is

$$\bar{r}_h = \lambda r_h, \quad \bar{T}_H = T_H / \lambda, \quad \bar{S} = \lambda^{m-1} S, \quad \bar{\mathcal{M}} = \lambda^{m-1} \mathcal{M}, \quad \text{and} \quad \bar{\mathcal{T}}_k = \lambda^{m-1} \mathcal{T}_k. \quad (4.15)$$

Then, given the full spectrum of solutions for $\ell = 1$ (with $0 < r_h < \infty$), one may find the corresponding branch for any value of $\Lambda < 0$.

We have constructed black hole solutions in all dimensions between five and ten with several values of m and for $0 \leq r_h \lesssim 10$. Thus they are likely to exist for any allowed set (d, m) and for any value of the event horizon r_h .

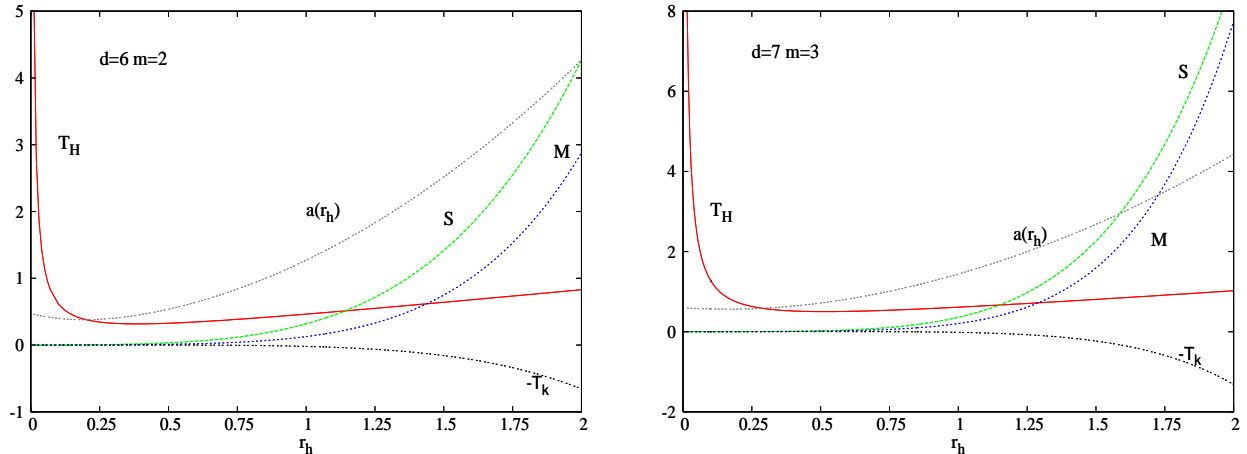


Figure 3: The mass-parameter M , the tension \mathcal{T}_k , the value of the metric function $a(r)$ at the event horizon as well as the Hawking temperature T_H and the entropy S of $m = 2, d = 6$ and $m = 3, d = 7$ black hole solutions are represented as functions of the event horizon radius.

As typical examples, the metric functions a, b and f are shown in Figure 2 as functions of the radial coordinate r for two values of (d, m) . One can see that the term r^2/ℓ^2 starts dominating the profiles of these functions very rapidly, which implies a small difference between them for large enough r .

The dependence of various physical parameters on the event horizon radius is presented in Figure 3 for $m = 2, d = 6$ and $m = 3, d = 7$ solutions. These plots exhibit the basic features of the solutions we found also in other dimensions and for other values of $m > 1$ (note that there and in Figure 4 we set $V_x \Omega_m / G = L_k = 1$ in the expressions for the mass, tension and entropy and we subtracted the constant Casimir terms in odd dimensions).

Similarly to the spherically symmetric Schwarzschild-AdS solutions, one can see in Figure 3 that the temperature of the black holes is bounded from below. At low temperatures we have a single bulk solution which we conjecture to correspond to the thermal globally regular soliton. At high temperatures there exist two additional solutions that correspond to the small and large black holes. For large black holes, the entropy is increasing with the temperature, while the small black holes have a negative specific heat.

Moreover, the free energy $F = I/\beta$ is positive for small r_h and negative for large r_h . This shows that the phase transition found in [40] occurs also in this case and there are two branches of solutions consisting of smaller (unstable) and large (stable) black holes. This is illustrated in Figure 4, where the free energy is plotted versus the temperature for $d = 7$ solutions with several values of m (the $m = 5$ configurations have $n = 1$ and correspond to Schwarzschild-AdS₇ black holes).

Without entering into details, we note that by performing a double analytic continuation, the black hole solutions in this work describe static bubbles of nothing in AdS, with a line

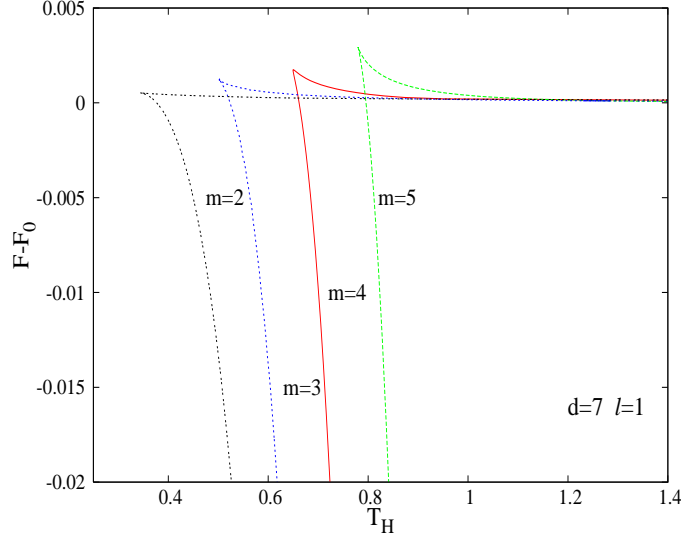


Figure 4: The free energy *vs.* the temperature for the small and large $d = 7$ black hole solutions is plotted for several values of m . Here we have subtracted the free energy contribution F_0 of the corresponding globally regular solutions.

element:

$$ds^2 = a(r)(-du^2 + \sum_{i,j=1}^{n-2} \delta_{i,j} dx^i dx^j) + b(r)d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\omega_m^2, \quad (4.16)$$

where τ has a periodicity $\beta = 1/T_H$. The properties of these solutions can be discussed in a similar way to the black hole case. For example, there are both ‘small’ and ‘large’ bubbles, which result as analytical continuation of the corresponding black hole branches. Using the counterterm approach, one can show that the mass of a bubble solution is

$$\mathcal{M}_{bubble} = -\beta \mathcal{T}_u. \quad (4.17)$$

Note also that the analytic continuation of a soliton leads to the same regular solution (since $a(r) = b(r)$ in that case), with an arbitrary value of β .

5. Further remarks. Conclusions

The purpose of this work was to present evidence for the existence of a new type of solutions of Einstein gravity with negative Λ . For such solutions, the topological structure of the boundary at infinity is the product of time and $S^m \times R^{d-m-2}$, with $m > 1$. Both globally regular, soliton-type solutions and black holes have been considered. Since we could not find exact solutions, we have resorted to numerical methods. Analytical expressions for the solutions can be constructed, however, close to the origin $r = 0$ (or to the event horizon $r = r_h > 0$) and for large values of r .

The solitons were used to construct new brane world models with compact extra dimensions. Different from the Randall-Sundrum [12] and the Karch-Randall models [45], the existence of extra-dimensions on the brane imposes the presence of matter fields, which have been taken to be topological solitons confined on the sphere S^m .

It is clear that this work has only scratched the surface of the full subject and a variety of asymptotically (locally) AdS solutions with more complex boundary structure are likely to be found. For example, as in the $n = 2$ case [14], a generalization of the black hole solutions in this work with the m -dimensional sphere $d\omega_m^2$ replaced by a hyperboloid $d\Xi_m^2$ should exist (note that these configuration will not possess a soliton limit). In fact, it would be interesting to study a more general class of black hole solutions with the line element

$$ds^2 = \frac{dr^2}{f(r)} + a(r) \sum_{i,j=1}^{n-1} \delta_{i,j} dx^i dx^j - b(r) dt^2 + P^2(r) d\omega_m^2 + c(r) d\Xi_p^2, \quad (5.1)$$

with $m + n + p + 1 = d$, the solutions in our paper corresponding to $p = 0$. The metric functions a, b, c and P would satisfy different boundary conditions at $r = r_h$ and thus would not be equal.

We did not address the question of classical stability of the new solutions in this work. For $n = 2$, the results in [17] show that the solutions are stable for large enough values of the event horizon radius only. We expect that the situation will be the same for any $n \geq 2$. This is suggested by the thermodynamical properties of the solutions, together with the Gubser-Mitra conjecture [47] that correlates the dynamical and thermodynamical stability for systems with translational symmetry and infinite extent. Therefore we expect the branch of black sole solutions with a negative specific heat to possess also a Gregory-Laflamme unstable mode.

In connection to that, it would be of particular interest to construct AdS black holes approaching the asymptotics (4.6), (4.7) as $r \rightarrow \infty$ but with a different topology of the horizon. For $n = 2$, these would be the AdS counterparts of the $\Lambda = 0$ caged black holes in Kaluza-Klein theory, see *e.g.* Ref. [46]. The existence of such configurations is suggested by the results in [17], [18].

Also, the configurations in this work can be used to construct new lower dimensional non-trivial soliton and black hole solutions of the Einstein-dilaton system with a Liouville dilaton potential. As with the $n = 2$ case in [14], these solutions are found by dimensionally reducing w.r.t. one (or several) Killing vector(s) $\partial/\partial x^i$. Moreover, by using the techniques in [14], one can show that the reduced action has an effective $SL(2, R)$ symmetry, which can be used to add an electric charge to these lower dimensional configurations.

We close this paper with several remarks on the possible role of the solutions in this work in the context of AdS/CFT correspondence. The background metric upon which the dual field theory resides is found by taking the rescaling $h_{ab} = \lim_{r \rightarrow \infty} \frac{\ell^2}{r^2} \gamma_{ab}$. Therefore, for both soliton and black hole solutions we find

$$ds^2 = h_{ab} dx^a dx^b = -dt^2 + \sum_{i,j=1}^{n-1} \delta_{i,j} dx^i dx^j + \ell^2 d\omega_m^2, \quad (5.2)$$

and so the conformal boundary, where the dual theory lives, is $R_t \times R^{n-1} \times S^m$.

The expectation value $\langle \tau_a^b \rangle$ of the stress tensor of the dual CFT can be computed using the relation [48]

$$\sqrt{-h}h^{ab} \langle \tau_{bc} \rangle = \lim_{r \rightarrow \infty} \sqrt{-\gamma} \gamma^{ab} T_{bc}, \quad (5.3)$$

where T_{bc} is the gravity boundary stress tensor (2.23).

Let us consider for example¹⁶ the (most interesting) case of black holes with a four dimensional flat subspace (*i.e.* $n = 4$). A straightforward computation gives the following expressions for the nonvanishing components of $\langle \tau_a^b \rangle$

$$\begin{aligned} \langle \tau_t^t \rangle &= \frac{1}{8\pi G} \frac{1 + 8000c_t - 4800c_z}{3200\ell}, \quad \langle \tau_x^x \rangle = \frac{1}{8\pi G} \frac{1 - 1600c_t + 4800c_z}{3200\ell}, \\ \langle \tau_\phi^\phi \rangle &= \frac{1}{8\pi G} \frac{5 - 800c_t - 2400c_z}{1600\ell}, \end{aligned} \quad (5.4)$$

for $m = 2$ (*i.e.* $d = 7$), and

$$\langle \tau_t^t \rangle = \frac{1}{8\pi G} \frac{6c_t - 3c_z}{2\ell}, \quad \langle \tau_x^x \rangle = -\frac{1}{8\pi G} \frac{c_t - 4c_z}{2\ell}, \quad \langle \tau_\phi^\phi \rangle = -\frac{1}{8\pi G} \frac{c_t + 3c_z}{2\ell}, \quad (5.5)$$

for $m = 3$ (*i.e.* $d = 8$). The stress tensor of the dual CFT defined on an eight dimensional space with a compact S^4 (*i.e.* a $d = 9$ bulk) is

$$\begin{aligned} \langle \tau_t^t \rangle &= -\frac{1}{8\pi G} \frac{221 - 12446784c_t + 7112448c_z}{3556224\ell}, \quad \langle \tau_x^x \rangle = -\frac{1}{8\pi G} \frac{221 - 12446784c_t + 7112448c_z}{3556224\ell}, \\ \langle \tau_\phi^\phi \rangle &= \frac{1}{8\pi G} \frac{875 - 5334336c_t - 21337344c_z}{10668672\ell}. \end{aligned} \quad (5.6)$$

In the above expressions, $\langle \tau_x^x \rangle$ and $\langle \tau_\phi^\phi \rangle$ stand for the nonvanishing components of the stress tensor of the dual CFT along the flat directions and on the sphere, respectively.

As expected, these stress tensors are finite and covariantly conserved. For even d , we have found that $\langle \tau_a^b \rangle$ is always traceless, as expected from the absence of conformal anomalies for the boundary field theory in odd dimensions. However, for odd d (*i.e.* an even dimensional boundary metric) $\langle \tau_a^b \rangle$ is *not* traceless. In fact, we have verified that for $d = 7$ its trace $\langle \tau_a^a \rangle = 3/(3200\pi G\ell)$ is precisely equal to the conformal anomaly of the boundary CFT in six dimensions [22]:

$$\mathcal{A} = -\frac{1}{8\pi G} \frac{\ell^5}{128} \left(RR^{ab}R_{ab} - \frac{3}{25}R^3 - 2R^{ab}R^{cd}R_{acbd} - \frac{1}{10}R^{ab}\nabla_a\nabla_b R + R^{ab}\square R_{ab} - \frac{1}{10}R\square R \right), \quad (5.7)$$

where R , R^{ab} and R_{abcd} are the curvature and the Ricci and Riemann tensor associated with the metric (5.2). A similar computation performed for the case $d = 5$, $m = 2$ leads to a boundary stress tensor whose trace matches precisely the conformal anomaly of the dual four-dimensional CFT [14].

Further analysis of these metrics and their role in string theory remain interesting issues to explore in the future.

¹⁶The expressions of $\langle \tau_a^b \rangle$ for $m = 2$, $d = 5$ and $m = 4$, $d = 7$ are given in [14], and [15], respectively.

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